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Ideal solution of an inverse normal mode problem
with finite spectral data.

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Summary. The problem of reconstructing the density of a vibrating string given the first N eigenfrequencies for two vibrating configurations admits an infinite number of solutions. Among all such strings compatible with the truncated data set, we define the ideal string to be that string for which a weighted average of the density is minimum. We prove that this ideal string must have a finite number of degrees of freedom and hence, that it is made up by a finite number of concentrated point masses. By specializing the optimality criterion, we can also show that the Krein string is an ideal string.

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1. Introduction.

This paper is concerned with the solution of inverse eigenvalue problems for which the data sets are insufficient for insuring unique solutions. This situation is typical of most inverse eigenvalue problems arising in Geophysics.

A very appealing approach to certain inverse problems with partial data sets was pioneered by Parker (1974, 1975). His original work dealt with inverse problems associated with the reconstruction of a buried body embedded in a known matrix from surface gravity anomaly data. Confronted with the inherent nonuniqueness of the problem, he decided to construct that particular body which was (i) capable of explaining the data and (ii) had the least possible maximum density. Parker called this unique body the ideal body. Thus, out of a set of many possible solutions, he singled out a particular one by means of an extremum criterion. This procedure yielded rigorous bounds for the entire class of solutions.

A similar approach has also proved very successful in studying certain turbulent flows (Howard 1963, 1972, Busse 1969). There the problem stems from the fact that the Navier-Stokes equations admit a multiplicity of solutions. Out of this class of solutions, a distinguished one is selected by means of an optimality criterion. This optimum solution can then yield general results, e.g. bounds on all solutions.

The present paper attempts to approach the simplest inverse normal modes problem, namely that for a vibrating string, from the same point of view. The data, which consist of truncated frequency spectra, cannot guarantee a unique solution to the inverse problem. Among the set of strings which have the same given eigenfrequencies, we shall select an ideal string. As a selection criterion, we shall minimize a weighted average of the density. We shall prove that this ideal string has a finite number of degrees of freedom and hence, is made up of concentrated point masses joined by weightless threads.

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The particular features of the ideal string can be found by solving a convex programming problem. Sabatier (1977a,b) was the first to point out the relationship between Parker's ideal body theory and the classical theory of linear programming. A similar relationship exists here. For a special ideality criterion, the solution to this programming problem can be trivially obtained. This special case is discussed at the end of the paper.

2. The extremum criterion for the ideal string.

The vibrating system of interest consists of a string of (dimensionless) density $\rho(x)$, of unit length and taut by a unit tension. Two vibrating configurations will be used, differing from each other by the fastening at the left end. The corresponding eigenvalue problems will be written thus:

$$\left. \begin{aligned} u_n'' + \lambda_n \rho u_n &= 0, \\ u_n(0) \cos \alpha - u_n'(0) \sin \alpha &= 0, \\ u_n(1) \cos \gamma + u_n'(1) \sin \gamma &= 0; \end{aligned} \right\} \quad (2.1)$$

and

$$\left. \begin{aligned} v_n'' + \mu_n \rho v_n &= 0, \\ v_n(0) \cos \beta - v_n'(0) \sin \beta &= 0, \\ v_n(1) \cos \gamma + v_n'(1) \sin \gamma &= 0. \end{aligned} \right\} \quad (2.2)$$

α, β, γ are parameters such that

$$0 \leq \alpha < \beta \leq \pi/2.$$

As a result the system represented by (2.1) is stiffer than that associated with (2.2) and the eigenfrequencies interlace as follows:

$$\mu_1 < \lambda_1 < \dots \quad \mu_n < \lambda_n < \dots \quad (2.3)$$

It is preferable to write (2.1) and (2.2) as integral equations, viz.

$$u_n(x) = \lambda_n \int_0^1 \rho(\xi) G(x, \xi) u_n(\xi) d\xi \quad (2.4)$$

and

$$v_n(x) = \mu_n \int_0^1 \rho(\xi) \Gamma(x, \xi) v_n(\xi) d\xi \quad (2.5)$$

where

$$G(x, \xi) = \frac{1}{\cos\alpha\cos\gamma + \sin(\alpha+\gamma)} \begin{cases} \{(1-\xi)\cos\gamma + \sin\gamma\}\{x\cos\alpha + \sin\alpha\} & x \leq \xi \\ \{(1-x)\cos\gamma + \sin\gamma\}\{\xi\cos\alpha + \sin\alpha\} & x > \xi \end{cases} \quad (2.6)$$

and $\Gamma(x, \xi)$ is obtained by replacing α by β in the above formula.

Given $\rho(x)$, (2.4) and (2.5) can be looked upon as recipes for calculating the corresponding eigenvalues λ_n and μ_n . For brevity, we denote the result of these calculations by the notations $\lambda_n[\rho]$ and $\mu_n[\rho]$ which emphasize that λ_n and μ_n are functionals of $\rho(x)$.

It is well known (Borg 1946), that the complete λ and μ spectra are necessary (and sufficient) for the unique determination of $\rho(x)$. Therefore, if we are only given the truncated spectra $\{\lambda_n\}_1^N$ and $\{\mu_n\}_1^N$ we cannot infer $\rho(x)$ uniquely: there are infinitely many strings with the first N natural frequencies. Let us denote the set of these strings by R_N , i.e.

$$R_N = \{\rho(x); \rho(x) \geq 0, \lambda_n[\rho] = \lambda_n, \mu_n[\rho] = \mu_n, n=1, 2, \dots, N\} \quad (2.7)$$

Next, we define a weighted average of $\rho(x)$, viz.

$$M[\rho] = \int_0^1 f(x) \rho(x) dx, \quad (2.8)$$

where $f(x)$ is a positive, continuously differentiable function of our own choosing.

The only other condition which we shall place on $f(x)$ is that:

$$f(1) \cos\gamma + f'(1) \sin\gamma = 1. \quad (2.9)$$

For instance, if $\gamma = 0$, we can take $f(x)$ to be equal to x^k , in which case $M[\rho]$ would correspond to the k -th moment of the density distribution.

We can now state the extremum criterion which will define the ideal string. From among all the strings in R_N , the ideal string $\hat{\rho}(x)$ is that string for which the weighted average M is a minimum.

3. Structure of the ideal string.

In order to find the density $\hat{\rho}(x)$ of the ideal string we must solve the following rather atypical problem in the calculus of variations.

$$\text{Minimize } M[\rho] = \int_0^1 f(x) \rho(x) dx \quad (3.1)$$

subject to the equality constraints

$$\left. \begin{aligned} \lambda_n[\rho] &= \lambda_n \\ \mu_n[\rho] &= \mu_n \end{aligned} \right\} (n = 1, 2, \dots, N) \quad (3.2)$$

and to the inequality constraint

$$\rho(x) \geq 0. \quad (3.3)$$

In order to transform this optimization problem into a more standard form, we shall first prove that the ideal string must be made up of a finite number J of concentrated point masses, i.e. that

$$\hat{\rho}(x) = \sum_{j=1}^J m_j \delta(x-x_j) \quad (3.4)$$

The proof is of the reductio ad absurdum type. Namely, if R_N^∞ denotes the subset of R_N which is made up of all the strings with an infinite number of eigenfrequencies,[†] then we shall assume that the ideal string is in that subset, i.e.

$$\hat{\rho}(x) \in R_N^\infty \quad (3.5)$$

and then see that this assumption leads to a contradiction.

We start by writing

$$\rho(x) = [r(x)]^2, \quad (3.6)$$

thus satisfying the inequality constraint (3.3). Since $\hat{\rho}^2(x) \in R_N^\infty$, we can solve (2.4) and (2.5) and construct the set of functions $\{\hat{w}_n(x)\}_1^\infty$ defined as follows:

[†]Note that this class of strings is not equivalent to the class $\rho \in L_1(0,1)$. Indeed, even though $\rho = 1 + m\delta(x-\frac{1}{2}) \notin L_1(0,1)$, it has an infinite number of eigenfrequencies. A rigorous treatment of this point would require the use of Stieltjes integrals.

$$\begin{aligned}\hat{w}_{2n-1}(x) &= [\hat{v}_n(x)]^2 \\ \hat{w}_{2n}(x) &= [\hat{u}_n(x)]^2\end{aligned}\quad n = 1, 2, \dots \quad (3.7)$$

This set of squares of the eigenfunctions is complete (Borg 1949, p. 61, Levitan 1952, 1964). This result is closely related to the theorem regarding the uniqueness of the solution of the inverse Sturm-Liouville problem. Heuristically, this result can be understood as follows. Consider two strings $\rho(x)$ and $\rho(x) + \delta\rho(x)$, where $\delta\rho$ is a small density variation. The corresponding eigenvalues differ by

$$\delta\lambda_n = - \frac{\int_0^1 \delta\rho(x) u_n^2(x) dx}{\int_0^1 \rho(x) u_n^2(x) dx} \quad (3.8a)$$

and

$$\delta\mu_n = - \frac{\int_0^1 \delta\rho(x) v_n^2(x) dx}{\int_0^1 \rho(x) v_n^2(x) dx} \quad (3.8b)$$

respectively. If it were possible to find a $\delta\rho(x) \not\equiv 0$, which is orthogonal to all the functions $\{w_n(x)\}_1^\infty$, then these two strings would have the same λ and μ spectra. But this is not possible on account of Borg's theorem (Borg 1946). Hence the functions $\{w_n(x)\}_1^\infty$, and in particular $\{\hat{w}_n(x)\}_1^\infty$, form a base.

Unfortunately, the base thus formed is not orthonormal. This is a minor nuisance since a second base, bi-orthogonal to the first is usually required. We denote this base by $\{\hat{\Omega}_n(x)\}_1^\infty$ and adopt the normalization

$$\int_0^1 \hat{w}_m(x) \hat{\Omega}_n(x) dx = \delta_{mn} \quad (3.9)$$

where δ_{mn} is the standard Kronecker delta. The actual construction of the functions $\hat{\Omega}_n(x)$ can be carried out by means of a Gram-Schmidt like procedure.

Let us consider next a string $[\hat{r}(x) + \delta r(x)]^2$ which is in R_N , i.e. a nearly ideal string. As can be seen from (3.8), the fact that this string is in R_N implies that $\hat{r}\delta r$ is orthogonal to the functions $\hat{W}_n(x)$ for $n = 1, 2, \dots, 2N$. Consequently,

$$\hat{r}\delta r = \sum_{n=2N+1}^{\infty} a_n \hat{\Omega}_n(x) \quad (3.10)$$

where the coefficients $\{a_n\}$ are arbitrary. Consequently, the variation in the weighted mean, viz.

$$\delta H = 2 \int_0^1 f(x) \hat{r} \delta r dx, \quad (3.11)$$

can be written thus:

$$\delta H = 2 \sum_{n=2N+1}^{\infty} a_n \int_0^1 f(x) \hat{\Omega}_n(x) dx. \quad (3.12)$$

Now, since $\hat{\rho}(x)$ is the ideal string, this variation must vanish for all a_n 's.

This requires that

$$\int_0^1 f(x) \hat{\Omega}_n(x) dx = 0 \quad n = 2N + 1, \dots \quad (3.13)$$

and therefore the function $f(x)$ admits the following finite series representation:

$$f(x) = \sum_{n=1}^{2N} f_n \hat{W}_n(x). \quad (3.14)$$

By means of simple manipulations, we can also write

$$f(x) \cos \gamma + f'(x) \sin \gamma = \sum_{n=1}^{2N} f_n [\hat{W}_n(x) \cos \gamma + \hat{W}'_n(x) \sin \gamma]. \quad (3.15)$$

Now, recalling condition 2.9) we can see that near $x = 1$, the left hand side of the above equation is a function nearly equal to 1, whereas on account of the boundary condition at $x = 1$, the right hand side represents a function nearly equal to 0. More specifically, we can always find an interval, say $(1-\epsilon, 1)$, over which (3.15) is false. We have reached a contradiction which implies that $\hat{\rho}(x)$ is not in R_N^∞ . Consequently $\hat{\rho}(x)$ must have a finite number of degrees of freedom and hence be of the form (3.4).

By making use of this knowledge about the structure of the ideal string, we can transform the original problem (3.1) - (3.3) into a programming problem. Indeed, by replacing (3.4) into (3.1) - (3.3), we can state the new problem as follows:

$$\text{Minimize } M = \sum_{j=1}^J f(x_j) m_j, \quad (3.15)$$

subject to the equality constraints

$$\left. \begin{aligned} G_n &\equiv \det(G_{ij}m_j - \frac{1}{\lambda_n} \delta_{ij}) = 0, \\ \Gamma_n &\equiv \det(\Gamma_{ij}m_j - \frac{1}{\mu_n} \delta_{ij}) = 0, \end{aligned} \right\} n = 1, 2, \dots, N \quad (3.16)$$

and the inequality constraints

$$\begin{aligned} 0 = x_0 &< x_1 < \dots < x_{J+1} = 1, \\ m_j &\geq 0, \quad j = 1, 2, \dots, J. \end{aligned} \quad (3.17)$$

In (3.16)

$$G_{ij} = G(x_i, x_j)$$

$$\Gamma_{ij} = \Gamma(x_i, x_j)$$

and "det" stands for determinant. Even though it is not possible to write the solution of (3.15) - (3.17) for a general function $f(x)$ and general spectral data, much is known about such problems (see e.g. Gass 1969).

4. A special case: the Krein string

If the function $f(x)$ is chosen as follows:

$$f(x) = \frac{1}{c} (x \cos \beta + \sin \beta)^2 \quad (4.1)$$

where c is determined by means of (2.9), viz.

$$c = (\cos \beta + \sin \beta) (\cos \beta \cos \gamma + \sin \beta \cos \gamma - 2 \cos \beta \sin \gamma) , \quad (4.2)$$

then the convex programming problem (3.15) - (3.17) can be solved very easily.

Note that for this case, the ideal string minimize a linear combination of the mass M_0 and the first and second moments M_1 and M_2 . Indeed,

$$M[\rho] = \frac{1}{c} (M_2 \cos^2 \beta + 2M_1 \sin \beta \cos \beta + M_0 \sin^2 \beta) . \quad (4.3)$$

It is possible to express this combination of M_0 , M_1 and M_2 in terms of the given eigenvalues, viz.

$$M_2 \cos^2 \beta + 2M_1 \sin \beta \cos \beta + M_0 \sin^2 \beta = \sin(\beta - \alpha) \frac{\cos \beta \cos \gamma + \sin(\beta + \gamma)}{\cos \alpha \cos \gamma + \sin(\alpha + \gamma)} \cdot \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{\infty} \left(1 - \frac{\mu_n}{\mu_k}\right)' \prod_{k=1}^{\infty} \left(1 - \frac{\mu_n}{\lambda_k}\right)} . \quad (4.4)$$

where the prime indicates that the term $k = n$ is omitted. This formula, which is derived in the appendix, generalizes the formula for M_0 first given by Krein (1951, 1952).[†]

Returning to the ideal string, we can show that the number of degrees of freedom of this string is equal to N . Indeed, if it were made up of $J > N$ point masses, then it would have J eigenfrequencies $\{\lambda_n\}_1^J$ and $\{\mu_n\}_1^J$, and

[†]Krein's formula is obtained by setting $\alpha = \gamma = 0$ and $\beta = \pi/2$.

$$M\left[\sum_{j=1}^J m_j \delta(x-x_j)\right] = Q \sum_{n=1}^J \frac{1}{\mu_n \prod_{k=1}^J \left(1 - \frac{\mu_n}{\mu_k}\right) \prod_{k=1}^J \left(1 - \frac{\mu_n}{\lambda_k}\right)} \quad (4.5)$$

where Q is a constant related to α , β and γ . Recalling that the μ and λ eigenvalues interlace, it is easy to see that all the terms in the above sum are positive. Consequently,

$$M\left[\sum_{j=1}^J m_j \delta(x-x_j)\right] \geq Q \sum_{n=1}^N \frac{P_n}{\mu_n \prod_{k=1}^N \left(1 - \frac{\mu_n}{\mu_k}\right) \prod_{k=1}^N \left(1 - \frac{\mu_n}{\lambda_k}\right)},$$

where

$$P_n = \prod_{k=N+1}^J \frac{1}{(1 - \mu_n/\mu_k)(1 - \mu_n/\lambda_k)} \quad (4.6)$$

On account of the ordering of the eigenvalues μ_n and λ_n , it is obvious that

$$P_n > 1. \quad (4.7)$$

As a result

$$M\left[\sum_{j=1}^J m_j \delta(x-x_j)\right] \geq M\left[\sum_{i=1}^N m_i' \delta(x-x_i')\right]. \quad (4.8)$$

In other words, given the truncated spectra $\{\lambda_n\}_1^N$ and $\{\mu_n\}_1^N$, the minimum of $M_2 \cos^2 \beta + 2M_1 \sin \beta \cos \beta + M_0 \sin^2 \beta$ is reached for a string with N -degrees of freedom. Since there is only one such string in R_N , this ideal string is uniquely determined. For the case $\alpha = \gamma = 0$, $\beta = \pi/2$ this string is none other than the Kravtsov string obtained by writing the rational fraction $\prod_{n=1}^N (1 - z/\lambda_n) / (1 - z/\mu_n)$ as a Stieltjes continued fraction, namely

$$\prod_{n=1}^N \frac{(1 - z/\lambda_n)}{(1 - z/\mu_n)} = \ell_0 + \frac{1}{-m_1 z + \frac{1}{\ell_1 + \frac{1}{\ddots + \frac{1}{\ell_N}}}}, \quad (4.9)$$

where $\ell_i = x_{i+1} - x_i$.

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Appendix

A generalization of Krein's formula.

We shall be concerned with a string of density $\rho(x)$ vibrating in two different configurations. The eigenvalue problem for the first configuration is:

$$\begin{aligned} u_n'' + \lambda_n \rho u_n &= 0 \\ u_n(0) \cos \alpha - u_n'(0) \sin \alpha &= 0 \end{aligned} \tag{A.1}$$

$$u_n(1) \cos \gamma + u_n'(1) \sin \gamma = 0$$

whereas the second is

$$\begin{aligned} v_n'' + \mu_n \rho v_n &= 0 \\ v_n(0) \cos \beta - v_n'(0) \sin \beta &= 0 \end{aligned} \tag{A.2}$$

$$v_n(1) \cos \gamma + v_n'(1) \sin \gamma = 0$$

Just as in the body of the paper, we can assume without loss of generality that $0 \leq \alpha < \beta \leq \frac{\pi}{2}$. As a result, system (A.1) is stiffer than system (A.2) and

$$\mu_1 < \lambda_1 < \dots < \mu_n < \lambda_n < \dots \tag{A.3}$$

In order to solve (A.1) and (A.2), we introduce two fundamental solutions $y_1(x; \lambda)$ and $y_2(x; \lambda)$ of the equation

$$y'' + \lambda \rho y = 0 \tag{A.4}$$

such that

$$\begin{aligned} y_1(1; \lambda) &= -\sin \gamma, \\ y_1'(1; \lambda) &= \cos \gamma, \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} y_2(1; \lambda) &= -\cos \gamma, \\ y_2'(1; \lambda) &= -\sin \gamma. \end{aligned} \tag{A.6}$$

In view of the linear independence of these solutions and of the conditions (A.5), (A.6) we can write

$$y_1(x;\lambda) y_2'(x;\lambda) - y_2(x;\lambda) y_1'(x;\lambda) = 1. \quad (\text{A.7})$$

It is convenient to introduce two pairs of auxiliary variables, namely

$$U(x;\lambda) = \cos \alpha y_1(x;\lambda) - \sin \alpha y_1'(x;\lambda) \quad (\text{A.8})$$

$$V(x;\lambda) = \cos \beta y_1(x;\lambda) - \sin \beta y_1'(x;\lambda)$$

and

$$\Phi(x;\lambda) = \cos \alpha y_2(x;\lambda) - \sin \alpha y_2'(x;\lambda) \quad (\text{A.9})$$

$$\Psi(x;\lambda) = \cos \beta y_2(x;\lambda) - \sin \beta y_2'(x;\lambda)$$

Several remarks are now in order. First of all, for $x = 0$ the zeros of U and V are the eigenvalues of (A.1) and (A.2). Consequently, in view of the fact that $U(0,\lambda)$ & $V(0,\lambda)$ are entire functions of λ of order $1/2$ (Titchmarsh 1962), we can write

$$U(0,\lambda) = U(0,0) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right), \quad (\text{A.10})$$

$$V(0,\lambda) = V(0,0) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right).$$

Next, we should point out that the Wronskian equality (A.7) can be written in terms of the new variables as follows:

$$U(x;\lambda) \Psi(x;\lambda) - V(x;\lambda) \Phi(x;\lambda) = \sin(\alpha - \beta). \quad (\text{A.11})$$

Let us now consider the function

$$F(Z;\lambda) = \frac{V(0;0) \Psi(0;Z) - \Psi(0;0) V(0;Z)}{Z(Z-\lambda) V(0;Z)} \quad (\text{A.12})$$

This is a meromorphic function with simple poles at $Z = \lambda$ and $Z = \mu_n$ ($n=1,2,\dots$) but not at $Z = 0$. If Λ_n is a circle in the Z -plane of radius $|Z| = \lambda_n$, then it is possible to show that

$$\lim_{n \rightarrow \infty} \left| \frac{1}{2\pi i} \oint_{\Lambda_n} F(z; \lambda) dz \right| = 0 \quad (\text{A.13})$$

Making use of the calculus of residues, we can rewrite (A.13) thus:

$$\frac{V(0;0)\Psi(0;\lambda) - \Psi(0;0)V(0;\lambda)}{\lambda V(0;\lambda)} = - \sum_{n=1}^{\infty} \frac{V(0;0)\Psi(0;\mu_n) - \Psi(0;0)V(0;\mu_n)}{\mu_n (\mu_n - \lambda) \left. \frac{\partial V}{\partial \lambda} \right|_{z=\mu_n}}$$

Several simplifications are possible. In particular, since $V(0;\mu_n) = 0$, we can exploit (A.11) to write $\Psi(0;\mu_n)$ in terms of $U(0;\mu_n)$; also we can use (A.10) to evaluate $\partial V(0,\lambda)/\partial \lambda$. Therefore

$$\frac{V(0;0)\Psi(0;\lambda) - \Psi(0;0)V(0;\lambda)}{\lambda V(0;\lambda)} = \sum_{n=1}^{\infty} \frac{\sin(\alpha-\beta)}{U(0;0)} \cdot \frac{1}{(\mu_n - \lambda) \prod_{k=1}^{\infty} \left(1 - \frac{\mu_n}{\mu_k}\right) \prod_{k=1}^{\infty} \left(1 - \frac{\mu_n}{\lambda_k}\right)}$$

So far, we have just repeated the various steps in the proof of the Mittag-Leffler theorem (Whittaker & Watson 1952). We now let $\lambda \rightarrow 0$. The above formula becomes:

$$\begin{aligned} & \frac{U(0;0)}{V(0;0)} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{ V(0;0)\Psi(0;\lambda) - \Psi(0;0)V(0;\lambda) \} \\ &= \sin(\alpha-\beta) \sum_{n=1}^{\infty} \frac{1}{\mu_n \prod_{k=1}^{\infty} \left(1 - \frac{\mu_n}{\mu_k}\right) \prod_{k=1}^{\infty} \left(1 - \frac{\mu_n}{\lambda_k}\right)} \end{aligned} \quad (\text{A.14})$$

We shall now see that the left hand side can be expressed in terms of the mass and first two moments of the density distribution. To that effect, let us expand $y_1(x;\lambda)$ in powers of λ and compute the first two terms by substituting in (A.1) & (A.2). Omitting the intermediary calculations, we get

$$\begin{aligned} y_1(0;\lambda) &= -\cos\gamma - \sin\gamma + \lambda[(M_1 - M_2)\cos\gamma + M_1\sin\gamma] + \dots \\ y_1'(0;\lambda) &= \cos\gamma + \lambda[-(M_0 - M_1)\cos\gamma - M_0\sin\gamma] + \dots \end{aligned} \quad (\text{A.15})$$

and

$$\begin{aligned} y_2(0;\lambda) &= \sin\gamma - \cos\gamma + \lambda[-(M_1 - M_2)\sin\gamma + M_1\cos\gamma] + \dots \\ y_2'(0;\lambda) &= \sin\gamma + \lambda[M_0 - M_1]\sin\gamma - M_0\cos\gamma + \dots \end{aligned} \quad (\text{A.16})$$

Replacing these expressions in (A.8), (A.9) we deduce that

$$\begin{aligned} V(0;\lambda) &= -[\cos\beta\cos\gamma + \sin(\beta+\gamma)] \\ &+ \lambda[(M_1 - M_2)\cos\beta\cos\gamma + M_1\cos\beta\sin\gamma \\ &+ (M_0 - M_1)\sin\beta\cos\gamma + M_0\sin\beta\sin\gamma] + \dots \end{aligned} \quad (\text{A.17})$$

and

$$\begin{aligned} \Psi(0;\lambda) &= [\cos\beta\sin\gamma - \cos(\beta+\gamma)] \\ &+ \lambda[-(M_1 - M_2)\cos\beta\sin\gamma + M_1\cos\beta\cos\gamma \\ &- (M_0 - M_1)\sin\beta\sin\gamma + M_0\sin\beta\cos\gamma] + \dots \end{aligned} \quad (\text{A.18})$$

and consequently

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{V(0;0)\Psi(0;\lambda) - \Psi(0;0)V(0;\lambda)\} &= (M_1 - M_2)\cos^2\beta \\ &- M_1(\cos\beta + \sin\beta)\cos\beta + (M_0 - M_1)\sin\beta\cos\beta \\ &- M_0(\cos\beta + \sin\beta)\sin\beta \end{aligned}$$

As a result (A.14) reads:

$$\begin{aligned} \frac{\cos\alpha\cos\gamma + \sin(\alpha+\gamma)}{\cos\beta\cos\gamma + \sin(\beta+\gamma)} (M_2\cos^2\beta + 2M_1\sin\beta\cos\beta + M_0\sin^2\beta) = \\ \sin(\beta-\alpha) \sum_{n=1}^{\infty} \frac{1}{\mu_n \prod_{k=1}^{\infty} (1 - \frac{\mu_n}{\mu_k}) \prod_{k=1}^{\infty} (1 - \frac{\mu_n}{\lambda_k})} \end{aligned} \quad (\text{A.19})$$

For $\alpha=0$, $\beta=\pi/2$ and $\gamma=0$, the formula reduces to

$$M_0 = \sum_{n=1}^{\infty} \frac{1}{\mu_n \prod_{k=1}^{\infty} (1 - \frac{\mu_n}{\mu_k}) \prod_{k=1}^{\infty} (1 - \frac{\mu_n}{\lambda_k})} \quad (\text{A.20})$$

which is the formula given by Krein (1952).